Comment on "nonlinear viscosity and Grad's method"

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(Received 7 November 2000; published 5 March 2002)

In their recent paper [Phys. Rev. E **60**, 4052 (1999)] Uribe and García-Colín suggest that the stress tensor associated with the nonlinear viscosity formula $\eta=\eta_0\sinh^{-1}\kappa/\kappa$ ($\kappa=$a Rayleigh dissipation function) vanishes asymptotically as the magnitude of the velocity gradient increases. In this Comment, it is pointed out that their remark is invalid, because the stress tensor asymptotically exhibits a logarithmic κ dependence. It is also pointed out that their evolution equations for the stress tensor components are missing the terms containing the velocity gradients in the transversal directions and, as a consequence, give rise to a vanishing shear stress, contrary to the experimental evidence of gas flow in a tube.

DOI: 10.1103/PhysRevE.65.033201 PACS number(s): 51.10.+y, 05.60.−k, 51.20.+d

In a recent paper [1] [Phys. Rev. E **60**, 4052 (1999)], henceforth referred to as Ref. [1] in this Comment, Uribe and García-Colín calculate nonlinear viscosity formulas of a dilute monatomic gas that undergoes a unidirectional flow. The kinetic equation used is the Boltzmann kinetic equation and the distribution function obeying the kinetic equation is assumed to have cylindrical symmetry. The unidirectional flow is parallel to the x axis of the coordinate system. The assumed cylindrical symmetry therefore makes the distribution function symmetric with respect to the y and z directions. Furthermore, the temperature is assumed to be uniform so that there is no heat flow. Because the flow is unidirectional in the x direction, the mean velocity $\mathbf{u}$ of the fluid has the x component only:

$$\mathbf{u}=(u_x,u_y,u_z)=(u_x,0,0). \tag{1}$$

The purpose of this Comment is to correct their comment on the non-Newtonian viscosity formula, Eq. (52) in Ref. [1], which I derived in my study of rheology on the basis of the Boltzmann equation, and to comment on the nonlinear viscosity obtained in their paper.

Having calculated the nonlinear viscosity, Uribe and García-Colín compare their viscosity formula with the aforementioned inverse hyperbolic sine formula that I have derived [2,3] by using the first-order cumulant approximation for the collision term in the Boltzmann equation:

$$\eta=\eta_0\frac{\sinh^{-1}\kappa}{\kappa}. \tag{2}$$

Here κ is related to the Rayleigh dissipation function and defined by

$$\kappa=[(mkT)^{1/4}\sqrt{\eta_0}/pd]([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}, \tag{3}$$

with η_0 denoting the Chapman-Enskog shear viscosity of the gas and d the diameter of the molecule and $[\nabla\mathbf{u}]^{(2)}=1/2[\nabla\mathbf{u}+(\nabla\mathbf{u})^t]-1/3\boldsymbol{\delta}\mathrm{Tr}(\nabla\mathbf{u})$, by now a well-recognized symbol in kinetic theory [4]. The symbol $\boldsymbol{\delta}$ is the unit second rank tensor. The viscosity formula (2) is valid under the approximation that neglects the normal stress differences $(P_{xx}-P_{yy})$ and $(P_{yy}-P_{zz})$ in the evolution equation for the stress tensor in the case of the unidirectional flow, as described, for example, in Ref. [3]. However, the formula (2) is consistent with the thermodynamic laws. It is a generic form of the Eyring formula [5] in rheology; the original Eyring formula, derived on the basis of the absolute reaction rate theory, contains an empirically adjustable parameter (relaxation time), whereas there is no adjustable parameter other than the potential parameters in the present expression. This viscosity formula has been rather extensively tested and shown to correctly account for a number of flow phenomena [6–13] in gases and liquids. It yields a vanishing nonlinear viscosity, as $([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}\to\infty$, and the Newtonian viscosity η_0, as $([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}\to 0$.

In their paper [1] Uribe and García-Colín suggest that this formula is probably incorrect in the case of implosion because the viscosity vanishes in the limit of $|\nabla_x u_x|\to\infty$ and thus the stress vanishes in the same limit. Their remark is not valid, because although the viscosity certainly vanishes in the aforementioned limit, the stress does not vanish, but *grows logarithmically*. This is easy to show, because, if we denote the norm of tensor $[\nabla\mathbf{u}]^{(2)}$ by $\|[\nabla\mathbf{u}]^{(2)}\|=\|[\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)}|^{1/2}$ and if there are no other gradient than, for example, $\nabla_x u_x$ in the flow in question as assumed in their argument against formula (2), then $\|[\nabla\mathbf{u}]^{(2)}\|=2/3|\nabla_x u_x|$ and for the xx component of the stress tensor we have the asymptotic form

$$|\Pi_{xx}|\equiv|P_{xx}-p|=\frac{2}{3}\eta_0\frac{\sinh^{-1}\kappa}{\kappa}|\nabla_x u_x|\sim\ln|\nabla_x u_x|, \tag{4}$$

as $|\nabla_x u_x|\to\infty$. Therefore their remark has no validity whatsoever. This form of the stress tensor certainly can distinguish implosion and explosion because then generally

$$\Pi_{xx}\sim\frac{[\nabla\mathbf{u}]^{(2)}_{xx}}{\|[\nabla\mathbf{u}]^{(2)}\|}\ln\|[\nabla\mathbf{u}]^{(2)}\|\sim\pm\ln\|[\nabla\mathbf{u}]^{(2)}\|,$$

which gives opposite signs for implosion and explosion.

I would like to make also the following comments. Before attempting comparison with other theories, their nonlinear viscosity formula should have been tested against some simulation or experimental results for nonlinear viscosity reported in the literature, as has been done for Eq. (2) since 1983 over a number of occasions [6–13]. Plausible limits of the nonlinear viscosity in special cases are by no means an assurance for its veracity in the face of experiment.

If the flow velocity is given by Eq. (1) and steady as is assumed in Ref. [1], then the steady-state equation of continuity is given by

$$\nabla_x(\rho u_x)=0, \tag{5}$$

where ρ is the mass density of the gas and $\nabla_x=\partial/\partial x$. This equation means that

$$\rho u_x = M, \tag{6}$$

where M is independent of x, but may depend on coordinates y and z, if the flow is not strictly one-dimensional. Unless the gas molecules are confined to move on a line parallel to the x axis, u_x will generally depend on y and z. Because of the distribution function taken, and the three-dimensional Boltzmann equation used for the calculation, the manner in which it is used in the calculation, and the fact that there appear the stress tensor components other than P_{xx} in the evolution equation for P_{xx} in Ref. [1], one is inevitably led to conclude that the flow considered is not really one-dimensional, but in fact, unidirectional, and hence, there is no reason to compel us to assume that u_x and the components of $\mathbf{P}$ do not depend on y and z. It must be emphasized that if the flow is one-dimensional, $\mathbf{P}$ has the P_{xx} component only, but no others in the case of their flow configuration. Since they consider other stress tensor components, $\nabla_y u_x=\nabla_z u_x=0$ is an additional assumption, which unfortunately yields deficient evolution equations for the stress tensor components derived in Ref. [1] and gives rise to a vanishing shear stress, contrary to the experimental evidence in the case of gas flow in a long tube. This assumption on the velocity gradients is not appropriate unless the continuum equations are truly one-dimensional and also the kinetic equation is one-dimensional, but the kinetic equation used is not one-dimensional. Since it follows from Eqs. (5) and (6) that $\nabla_x u_x = M(y,z)\nabla_x v$, where $v=1/\rho$ is the specific volume and $\nabla_x v$ may be taken for a constant if the rate of volume change is a constant, it obviously suggests that the y and z dependence of stress tensor components P_{xx}, P_{yy}, P_{xy}, etc. appearing in their stress evolution equations cannot be ruled out for the flow problem.

Another point about their nonlinear viscosity formula is that it predicts two different material functions (viscosities) depending on the sign of the velocity gradient, namely, implosion and explosion of the same gas have two different nonlinear viscosity, values. If nonlinear viscosities are material functions characteristic of a given material, then it is questionable why the gas should have two different material functions depending on whether it contracts or expands. What really counts in explaining flow phenomena is not the nonlinear viscosity, but the stress tensor appearing in the hydrodynamic equations, that is, the momentum and energy balance equations. Preoccupation with a nonlinear viscosity in the face of flow phenomena far removed from equilibrium can be misleading with regard to the flow problem in hand.

In conclusion, the vanishing nonlinear viscosity in Eq. (2) does not mean that the stress tensor also vanishes in the limit of $|\nabla_x u_x|\to\infty$; on the contrary, the stress tensor in fact grows logarithmically with $|\nabla_x u_x|$. Therefore the statement of Uribe and García-Colín regarding the asymptotic behavior of the stress tensor in connection with Eq. (2) is incorrect. The form of the distribution function and the manner in which it is used, the kinetic equation, and the stress evolution equations for various stress tensor components in Ref. [1] suggest that the velocity gradients $\nabla_y u_x$ and $\nabla_z u_x$, which were set equal to zero because of the assumption that u_x is a function of x only, are unjustifiably absent in the the stress evolution equations derived by Uribe and García-Colín for the unidirectional flow. Because of this assumption, the shear viscosity turns out to be absent and the stress tensor evolution equation has become inappropriate for the flow problem considered. If the flow is truly one-dimensional, the stress tensor components P_{yy}, P_{xy}, and /or P_{yz} should not appear in the evolution equation for P_{xx}.

This work was supported in part by the Natural Sciences and Engineering Research Council and the FCAR through the Center for the Physics of Materials, McGill University.

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